SEPARABILITY PROPERTIES OF ALMOST — DISJOINT FAMILIES OF SETS

BY

PAUL ERDÖS AND SAHARON SHELAH

ABSTRACT

We solve here some problems arising from a work by Hechler [3]. We eliminate extra set-theoretic axioms (MA, in fact) from existence theorems and deal with the existence of disjoint sets.

Intrdouction

We deal with almost-disjoint families (denoted by K and L) of sets of natural numbers. Usually the sets and the family are infinite. (For any two cardinals $\aleph_\beta \leqq \aleph_\alpha$, it is of interest to consider those families of subsets of $\aleph_\alpha$ such that each member of the family has cardinality $\aleph_\alpha$ and the intersection of any two distinct members of the family has cardinality less than $\aleph_\beta$. Our results generalize to hold for such families with only small changes or additional requirements (e.g., $\beta < \alpha$ or $\aleph_\alpha$ regular for Theorem 2.1).) We use Hechler's notation. Two remarks are in order:

1) non-2-separability of K is equivalent to the property (B) of K (see Miller [5] and Erdös and Hajnal [1] concerning this property). Miller proved the existence of, what we called, the 2-separable family in a very "tricky" way.

2) K is n-separable iff it does not have a colouring with n-colours (according to the notations of Erdös and Hajnal [2]).

1. Existence of n-separable but not $(n+1)$ — separable families

In [3], section 8, Hechler proves the existence of some almost-disjoint families with separability properties, using the assumption that every infinite maximal almost-disjoint family ($\subseteq P(N)$) has power $2^{\aleph_0}$. This follows from Martin's axiom

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[4] but, by Hechler [6], its negation is consistent with ZFC. We shall eliminate this assumption of [3], theorems 8.1 and 8.3.

THEOREM 1.1. *There is a strongly n-separable (hence n*-separable), non-$(n+1)$-separable, and even non-$(n+1)$-*separable, maximal almost-disjoint family (for any $n \geqq 1$).*

PROOF. Let $(A_1, \cdots, A_{n+1})$ be a partition of N into $(n+1)$ infinite sets. Let, for $i \leqq n+1$, $L_i = \{F^i_\alpha : \alpha < 2^{\aleph_0}\}$ be an almost-disjoint family of (infinite) subsets of A_i. (Throughout this paper we shall use i, j, k, m, and n to denote positive integers or variables ranging over positive integers. Thus $i \leqq n$ may always be thought of as meaning $1 \leqq i \leqq n$.) Let $\{(D^1_\alpha, \cdots, D^n_\alpha) : \alpha < 2^{\aleph_0}\}$ be the set of all partitions of N into n sets, each partition appearing $2^{\aleph_0}$ times. For each $\alpha < 2^{\aleph_0}$ and each $i \leqq n+1$,

$$F^i_\alpha = \bigcup_{j=1}^{n} (F^i_\alpha \cap D^j_\alpha)$$

Since F^i_α is infinite, there exists a $j = j(\alpha, i)$ such that $F^i_\alpha \cap D^j_\alpha$ is infinite. Since for fixed α the function $j(\alpha, i)$ has $n+1$ elements in its domain and only n in its range, there exist $i(\alpha,1) < i(\alpha,2) \leqq n+1$ such that $j(\alpha, i(\alpha,1)) = j(\alpha, i(\alpha,2)) \overset{df}{=} j(\alpha)$. Define $G_\alpha = D^{j(\alpha)}_\alpha \cap (F^{i(\alpha,1)}_\alpha \cup F^{i(\alpha,2)}_\alpha)$.

Let $K = \{G_\alpha : \alpha < 2^{\aleph_0}\}$. K is a subfamily of the desired family. Clearly it is an infinite almost-disjoint family of subsets of N. The partition $(A_1, \cdots, A_{n+1})$ shows that K is not even $(n+1)$-*separable, much less $(n+1)$-separable because each G_α intersects at least two A_i's in an infinite set. On the other hand, as $G_\alpha \subseteq D^{j(\alpha)}_\alpha$, and each partition appears infinitely often, K is strongly n-separable. Now, by [3] theorem 6.2, we may extend K to a maximal almost-disjoint family which retains these properties.

THEOREM 1.2. *For each $n > 1$, there is an n-separable maximal almost-disjoint family which is not strongly n-separable.*

PROOF. Let $(A_1, \cdots, A_n)$ be a partition of N into n infinite sets. For each $i \leqq n$, let $L_i = \{F^i_\alpha : \alpha < 2^{\aleph_0}\}$ be an almost-disjoint family of infinite subsets of A_i. Let $\{(D^1_\alpha, \cdots, D^n_\alpha) : 0 < \alpha < 2^{\aleph_0}\}$ be the set of partitions of N into n sets. We define for each $\alpha < 2^{\aleph_0}$, a set $G_\alpha \subseteq N$, and then $K = \{F^1_0, \cdots, F^n_0\} \cup \{G_\alpha : 0 < \alpha < 2^{\aleph_0}\}$ is our family. The partition $(A_1, \cdots, A_n)$ shows that K is not strongly n-separable; whereas the G_α's show that it is n-separable.

Let $0 < \alpha < 2^{\aleph_0}$. As in Theorem 1.1, for each $i \leqq n$, there is a $j = j(i,\alpha)$ such that $|F_\alpha^i \cap D_\alpha^j| = \aleph_0$. If there exists $i < k \leqq n$ such that $j = j(i,\alpha) = j(k,\alpha)$, then set $G_\alpha = D_\alpha^j \cap (F_\alpha^i \cap F_\alpha^k)$. Otherwise for each $j \leqq n$, there is an $i(j,\alpha)$ such that $i = i(j,\alpha) \Leftrightarrow j = j(i,\alpha)$. If there is a $k \leqq n$ such that $D_\alpha^k \nsubseteq A_{i(k,\alpha)}$, choose such a k and any $x \in D_\alpha^k - A_{i(k,\alpha)}$ and let $G_\alpha = (D_\alpha^k \cup F_\alpha^{i(k,\alpha)}) \cap \{x\}$. In the remaining case $D_\alpha^k = A_{i(k,\alpha)}$ for all k so the partitions $(D_\alpha^1, \cdots, D_\alpha^n)$ and $(A_1, \cdots, A_n)$ are the same and we may let $G_\alpha = F_0^1$. Clearly we obtain a family K satisfying our conditions.

Problem A. Does there exist a completely separable family (without assuming MA, as in [3], theorem 8.2)?

Problem B. For any $m, n \geqq 2$, does there exist an m-n-separable but not strongly m-n-separable almost-disjoint family? (For definition see [3], p. 415.)

Problem C. For any $m, n \geqq 2$ does there exist a strongly m-n-separable, non-m-$(n+1)$-separable almost-disjoint family?

Problem D. Let $m \geqq 1$. Does there exist am almost-disjoint family K, which is m-n-separable for every n, but is not $(m+1)$-2-separable?

Problem E. Does there exist a fully-Ramsay, not completely separable almost-disjoint family (see [3] p. 419)? The answer is no since if S is fully-Ramsey, $2S = \{\{2n : n \in A\} : A \in S\}$ is a counter-example.

REMARK. In Erdös and Hajnal [1], it was noted that Miller's [5] construction gives somewhat more than almost-disjointness, i.e., for each $A \in K$ and $x \in N - A$, the set $A \cap (\cup \{B; x \in B \in K\})$ is finite; with small additions our proofs can give this too. Notice that $CH(2^{\aleph_0} = \aleph_1)$ implies MA.

2. On disjoint sets in 2-separable almost-disjoint families

In [3], theorem 4.1, Hechler proved that any strongly 2-separable almost disjoint family contains an infinite disjoint subfamily. For 2-separability he has some weaker results (theorems 4.3 and 8.4). We shall prove that every such family has two disjoint sets, but (assuming MA)) not necessarily three.

THEOREM 2.1. *If K is an almost-disjoint 2-separable family of infinite sets, then it contains two disjoint sets.*

REMARK. We need the "infinite sets". For example

$$K_n = \{A : A \subset \{1, \cdots, 2n+1 \mid |A|| = n+1\}.$$

PROOF. Suppose there are no two disjoint sets in K. Let $A \in K$. We now define by induction on n a family $\{B_n\} \subseteq K - \{A\}$ of distinct sets and a colouring of the points of $\bigcup_{i=1}^{n} B_i$ by red and blue, such that each set B_{2n} contains only blue points except for one red point $y_{2n} \in B_{2n} \cap A$, and each set B_{2n+1} contains only red points except for one blue point $y_{2n+1} \in B_{2n+1} \cap A$. Suppose $B_1, \cdots, B_{n-1}$ have already been defined, together with the associated colouring. We shall define B_n assuming, without loss of generality, that n is even. Choose blue points $x_i \in B_i$ for each $i \leqq n-1$. Let $C = \{x_i: 1 \leqq i \leqq n-1\} \cup \bigcup_{i=1}^{n-1} (A \cap B_i)$. Since K is almost disjoint, C is finite. $(C, N-C)$ is a 2-partition of N, but since C is finite, no subset of it belongs to K. Hence there is a set $D \in K$ such that $D \subseteq (N-C)$. By assumption D and A are not disjoint, so choose any point $y_n \in A \cap D$. Then $y_n \notin C$, and as $y_n \in A$, we have $y_n \notin \bigcup_{i=1}^{n-1} B_i$. Let $D_1 = (\bigcup_{i=1}^{n-1} B_i \cup D) - \{x_1, \cdots, x_{n-1}, y_n\}$. As $x_i \in B_i$, we have $B_i \nsubseteq D_1$, and as $y_n \in D$, we also have $D \nsubseteq D_1$. If for any other set $X \in K$, we have $X \subseteq D_1$, then either $X \cap B_i$ (for some i) or $X \cap D$ is infinite—a contradiction.

Thus no member of K is contained in D_1. As K is 2-separable, there is a $B_n \in K$, such that $B_n \subseteq (N - D_1)$. By assumption $B_n \cap D \neq 0$, but by the definition of D_1 and B_n we have $(B_n \cap D) \subseteq \{y_n\}$. Hence $y_n \in B_n$. Similarly

$$B_n \cap \left(\bigcup_{i=1}^{n-1} B_i\right) \supseteq \{x_1, \cdots, x_n\}.$$

So all the points of B_n which are coloured, are coloured blue. Thus since $y_n \notin \bigcup_{i=1}^{n-1} B_i$, it is not coloured. So we can colour y_n red and each $x \in B_n - \{y_n\}$ blue. After we finish colouring $\bigcup_{n=1} B_n$, we can complete the colouring arbitrarily.

Now we have a partition of N into two sets—the red points and the blue points. Then one of them, say the set of red points, contains an $X \in K$. Now by assumption, for each n, $X \cap B_n \neq \emptyset$. But if n is even, B_n has only one red point y_n so $y_n \in X$. Hence $X \cap A \supseteq \{y_n \mid, n \text{ even}\}$ which is infinite—a contradiction.

THEOREM 2.2. *Assuming Martin's axiom, there is an (infinite) almost-disjoint 2-separable family of (infinite) subsets of N, containing no three disjoint sets.*

PROOF. Let $\{(D_\alpha^1, D_\alpha^2): \omega < \alpha < 2^{\aleph_0}\}$ be the set of partitions of N into two sets such that $0 \in D_\alpha^1$.

We shall define by induction on α a family of (infinite) sets $G_\alpha \subseteq N$ such that

1) N minus any finite union of G_α's is infinite.

2) $\beta < \alpha$ implies $G_\beta \cap G_\alpha$ is finite or $G_\beta = G_\alpha$.

3) $G_\alpha \subseteq D_\alpha^1$ or $G_\alpha \subseteq D_\alpha^2$.

4) If $\beta < \alpha, G_\alpha \neq G_\beta$, then either $0 \in G_\alpha \subseteq D_\alpha^1$ or $G_\alpha \cap G_\beta \neq \emptyset$.

Define G_n, $n < \omega$, so that $\{G_n : n < \omega\}$ is an almost-disjoint family of subsets of N with intersection $\{0\}$ and union N.

Suppose we have defined G_β for every $\beta < \alpha$, and we want to define G_α.

Case I. There exist $n, \beta_1 < \cdots < \beta_n < \alpha$, such that $D_\alpha^1 \subseteq^* \bigcup_{i=1}^n G_{\beta_i}$. ($A \subseteq^* B$ iff $A - B$ is finite).

If for some $\beta < \alpha$ we have $G_\beta \subseteq D_\alpha^1$, let $G_\alpha = G_\beta$. Clearly the conditions are satisfied. Otherwise, for each $G_\beta \notin \{G_{\beta_i} : i \leqq n\}$, condition 2 guarantees that $G_\beta \cap D_\alpha^1$ is finite and hence $G_\beta \cap D_\alpha^2$ is infinite. By [3] theorem 9.2, there is a set $A \subseteq D_\alpha^2$ which is almost disjoint to every $G_\beta \cap D_\alpha^2$, and $|G_\beta \cap D_\alpha^2| \geqq \aleph_0 \Rightarrow |G_\beta \cap A| > 0$ and $A \cap G_{\beta_i} \neq \emptyset$ for $1 \leqq i \leqq n$. Define $G_\alpha = A$; clearly all conditions are satisfied.

Case II. not case I.

By [3], section 9.2, we can find $A \subset D_\alpha^1$ such that A is infinite and $A \cap G_\beta$ finite for every $\beta < \alpha$. Let $G_\alpha = A \cup \{0\}$. The family $K = \{G_\alpha : \alpha < 2^{\aleph_0}\}$ satisfies all conditions except maximality. By [3], theorem 2.3, there is a $L \supset K$ which satisfies them all if we add 0 to every $A \in L - K$.

Problem F. Can Martin's axiom be eliminated from the proof?

REMARK. Clearly in Theorem 2.1, the "almost-disjoint" assumption was necessary (e.g., any ultrafilter over N is 2-separable, but it contains no two disjoint sets.) It is natural to ask whether the "almost-disjoint" hypothesis can be replaced by a weaker one. A natural candidate is given by:

DEFINITION 2.1. A family of sets is independent if for no n and distinct $A, B_1, \cdots, B_n$ in the family, $A \subseteq \bigcup_{i=1}^n B_i$.

If we replace $A \subseteq \cup B_i$ by $A \subseteq^* \cup B_i$ ($= A - \cup B_i$ is finite) we get the notion of *-independent. When considering a *-independent family, it is natural to ask as to whether or not it contains an almost-disjoint subfamily.

THEOREM 2.3. *Assuming Martin's axiom, there is an *-independent (infinite) strongly 2*-separable family K of (infinite) subsets of N, in which there are no two *-disjoint sets* (i.e., $A \neq B \in K \Rightarrow A \cap B$ *is infinite*).

PROOF. Let $\{(D_\alpha^1, D_\alpha^2) : \alpha < 2^{\aleph_0}\}$ be the set of partition of N into two, each appearing $2^{\aleph_0}$ times. We define by induction on α, infinite sets $G_\alpha \subseteq N$ such that:

1) for no $n, \beta_1, \cdots, \beta_n \leqq \alpha$, $N \nsubseteq^* \bigcup_{i=1}^n G_{\beta_i}$

2) $\beta < \alpha$ implies $G_\beta \cap G_\alpha$ is infinite

3) $\{G_\beta : \beta \leqq \alpha\}$ is *-independent.

Suppose G_β, $\beta < \alpha$, has been defined. Then clearly by 3) $\{G_\beta : \beta < \alpha\}$ is a *-independent family.

If for some $\beta < \alpha$, $G_\beta \subseteq^* D_\alpha^1$ or $G_\beta \subseteq^* D_\alpha^2$, let $G_\alpha = G_\beta$. Otherwise for each $\beta < \alpha$, $G_\beta \cap D_\alpha^2$ is infinite. By 1), without loss of generality, for no $n < \omega$, $\beta_1, \cdots, \beta_n < \alpha$, $D_\alpha^2 \subseteq^* \bigcup_{i=1}^n G_{\beta_i}$. Let L be the Boolean algebra generated by $\{G_\beta \cap D_\alpha^2 : \beta < \alpha\}$. Then $|L| < 2^{\aleph_0}$. Hence by Martin's axiom (see [3], theorem 9.2) we can find $G_\alpha \subseteq D_\alpha^2$ such that $A \in L$, A infinite $\rightarrow G_\alpha \cap A$ and $A - G_\alpha$ are infinite. So it is easy to verify that the induction hypothesis is satisfied. $K = \{G_\alpha : \alpha < 2^{\aleph_0}\}$ is the set we want.

Problem G. Does every independent 2-separable family of infinite subsets of N contain two disjoint members?

Problem G was solved affirmatively by Hajnal, McKenzie and Shelah, independently.

THEOEREM 2.4. *In every independent 2-separable family of infinite subsets of N, there are two disjoint sets.*

SKETCHED PROOF. Suppose K is a counterexample. Let

$$K_1 = \{A : A \in K,\ A \subseteq \ \cup \{B : B \in K,\ B \neq A\}\};$$

K_1 is also an independent 2-separable family. Define inductively $B_n \in K_1$, $x_n \in B_n$, and a colouring of $\bigcup_{i \leqq n} B_i$ by red and blue such that: x_n is the only red or blue point of B_n; and for each $x \in B_n$ there is $m < \omega$ such that $x = x_n$. Suppose x_i, B_i $i < n$, and the colouring of $\bigcup_{i<n} B_i$ are defined. Let x_n be the first number in $\bigcup_{i<n} B_i - \{x_i : i < n\}$, and, without loss of generality, x_n is blue. We want to find B_n and a colouring. Choose from each B_i, $i < n$, $x_n \notin B_i$, a red point z_i. Let $D_1 = \bigcup_{i<n} B_i - \{z_i : i\} - \{x_n\}$ and $D_2 = N - D_1$. For no $B \in K_1$ is $B \subseteq D_1$ so there is a $B_n \in K$ such that $B_n \subseteq D_2$. Colour $B_n - \{x_n\}$ by red.

By the 2-separability there is a set $B \in K_1$, disjoint to one colour, e.g., red. Hence if x_n is blue, $B \cap B_n \subseteq \{x_n\}$ so $B \cap B_n = \{x_n\}$ and $x_n \in B$. So B contains all the blue x_n. Let x_m be red. Then $B_m - B = \{x_m\}$, but $B_m \in K_1$, so we have $B' \in K$, $B' \neq B_m$ and $x_m \in B'$. Hence $B_m \subseteq B \cup B'$—a contradiction.

We can pose instead:

Conjecture G^*.

1) For every n there is a 2-separable family K of infinite subsets of N, with no two disjoint members, such that for distinct $B, A_1, \cdots, A_n \in K$, $B \nsubseteq \bigcup_i A_i$.

2) The same as 1) with $B \nsubseteq^* \bigcup_{i \leqq n} A_i$.

For $n = 1$, 1) was proved by Lovan (private communication) and Shelah independently.

A variant of Lovan's construction is: let us partition N into the infinite sets X, A_n, $n < \omega$. Let $\{T_\alpha : \alpha < 2^{\aleph_0}\} = \{T: T \subseteq \bigcup_n A_n, |T \cap A_n| = 1\}$, $X = \{x_n : n\} \cup \{y\}$, and $K = \{A_n \cup \{y\}: n < \omega\} \cup \{T_\alpha \cup \{y\}: \alpha < 2^{\aleph_0}\} \cup \{A_n \cup T_\alpha \cup \{x_m\}: n, m < \aleph, \alpha < 2^{\aleph_0}\} \cup \{X\}$.

Shelah's construction defines an increasing sequence of families K_α, such that $x \in A \in K_\alpha \Rightarrow (N - A) \cup \{x\} \in K_\alpha$.

3. Families of finite sets

There are also related finite problems. Let n, m be natural numbers. A family S is called an (n, m)-family if $A \in S$ implies $|A| = n$, and for distinct $A, B \in S$, we have $|A \cap B| \leqq m$. The question is to find $f(n, m)$ according to:

Definition. 3.1. $f(n, m)$ is defined to be the maximal number f such that every 2-separable (n, m)-family has in it f pairwise disjoint members.

For simplicity we restrict ourselves to $m = 1$.

Conjecture H. $f(n, 1) \geqq 2^{(n/2)(1-\varepsilon)}$ for any $\varepsilon > 0$ n big enough, (or at least $f(n, 1) \geqq 2^{cn}$).

However it is not hard to see that for n sufficiently large we have $f(n, 1) \geqq 2$ (and, in fact, much larger).

Suppose there are no two disjoint sets in a 2-separable $(n, 1)$-family S. Choose $x_0 \in A_0 \in S$ and let $V = \cup \{A : A \in S\}$. Let $B_0 = \cup \{A : x_0 \in A \in S\}$, and consider the partition $[B_0 - \{x_0\}, (V - B_0) \cup \{x_0\}]$. If some $C \in S$ is a subset of $(V - B_0) \cup \{x_0\}$, then $C \nsubseteq B_0$. Hence $x_0 \notin C$ so $C \subset V - B_0$ and therefore $C, A_0 \in S$ are disjoint—a contradiction. Hence there is a $C \in S$ such that $C \subset B_0 - \{x_0\}$. For each A if $x_0 \in A \in S$, $C \cap A \neq \emptyset$; but for any distinct $A_1, A_2 \in S$, $x_0 \in A_1$, $x_0 \in A_2$, $C \cap A_1 \cap A_2 = \emptyset$ as $|A_1 \cap A_2| \leqq 1$. Hence $A_1 \cap A_2 = \{x_0\}$ but $x_0 \notin C$. As $|C| = n$, clearly $|\{A : x_0 \in A \in S\}| \leqq n$. If $x_0 \in A_1 \in S$, $x_0 \in A_2 \in S$, $A_1 \neq A_2$ then for every $x_0 \neq x_1 \in A_1$, $x_0 \neq x_2 \in A_2$, there is at most one $C \in S$ such that $x_0 \notin C$, $x_1 \in C$ and $x_2 \in C$. Hence $|S| \leqq n + (n-1)^2$.

But we could have chosen x_0 belonging to at least two members of S; otherwise S is a family of pairwise disjoint sets, and, if $n > 1$, is not necessarily 2-separable.

Now we shall show that the 2-separability of S implies $|S| \geqq 2^{n+1}$ (in fact $|S| \geqq 2^n(1 + 2/n)^{-1}$ by Schmidt [8] and for $n = 4$ Prevling (private communication) has shown $|S| > 13$). We do it by a probabilistic argument. Suppose we randomly partition V into two parts such that each element of V has equal probability of falling into either part and that the choices are made independently. The probability of a set $A \in S$ being totally in one part is $2^{-(n-2)}$. But if $|S| < 2^{-n-1}$ the probability that at least one set of S will be totally in one part is at most $|S| \cdot 2^{-(n-1)} < 1$, so S cannot be 2-separable.

Thus we have shown that $2^{n-1} \leqq S \leqq n + (n-1)^2$. But $n + (n-1)^2 \geqq 2^{n-1}$ implies $n < 6$, so we have therefore shown that $n \geqq 6$ implies $f(n,1) \geqq 2$.

Erdös [7] shows that there is a family S such that $A \in S \Rightarrow |A| = n$, S is 2-separable and $|S| < cn^2 2^n$.

Conjecture I. For every n, $m(n) \geqq cn$ where $m(n)$ is the largest m for which $f(n,m) > 1$. (But $m(n) \geqq cn/\log n$ can be proved.)

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